Groups and nonlinear dynamical systems

Dynamics on the $SU(2)$ group

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Abstract: An abstract Newton-like equation on a general Lie algebra is introduced such that orbits of the Lie-group action are attracting set. This equation generates the nonlinear dynamical system satisfied by the group parameters having an attractor coinciding with the orbit. The periodic solutions of the abstract equation on a Lie algebra are discussed. The particular case of the $SU(2)$ group is investigated. The resulting nonlinear second-order dynamical system in $\mathbf{R}^3$ as well as its constrained version referring to the generalized spherical pendulum are shown to exhibit global Hopf bifurcation.

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Over the years, a variety of the group-theoretic methods for the study of differential equations have been devised [1,2]. It is scarcely to belabor their importance in the investigation of integrable systems. Let us only recall the methods for constructing group-invariant solutions. Nevertheless, there is still a general lack of approaches discussing behaviour of nonintegrable equations. Recently, an interesting approach was proposed by Okniński [3] relying on the study of nonlinear recurrences by asscociating them with a discrete-time evolution on a Lie group. More precisely, within the Okniński approach the use is made of the Shimizu-Leutbecher sequence [4,5] which satisfies the following group recurrence:

$$g_{k+1} = g_k h g_k^{-1}, \tag{1.1}$$

where g_k's and h are elements of a Lie group G. On taking a concrete realization of a Lie group G we arrive at the system of nonlinear difference equations which obey the group parameters. For example in the case with the $SU(2)$ group one obtains the logistic equation satisfied by one of the coordinates of the system implied by (1.1). By choosing $G = E(2)$ and $G = SU(2)$ for the Schimizu-Leutbecher sequence (1.1) the method was applied to the study of random walk on the plane [3] and discrete symmetries in chaos-order transitions [6], respectively. The question naturally arises as to whether such approach can be generalized to involve the case with continuous-time equations. An attempt in this direction was made by Okniński [7] who studied the equation with delayed argument arising from (1.1) by setting k to be a continuous parameter.

In this paper we introduce an abstract Newton-like equation on a general Lie algebra which can be regarded as a continuous-time algebraic version of (1.1), such that the orbits of the Lie-group action are attractors. The corresponding group parameters fixing a concrete algebra obey the system of nonlinear second-order differential equations having a limit set coinciding with the orbit. In section 2 we introduce the abstract Newton-like equation on a Lie algebra and study its asymptotic behaviour. We show that orbits of the Lie-group action are attracting set. Section 3 discusses periodic solutions to the abstract Newton-like equation. Namely, we establish some criteria of the existence of such solutions. In section 4 we illustrate the theory by an example of the $SU(2)$ group. We first discuss the nonlinear dynamical system generated by the corresponding Newton-like equation on the $su(2)$ algebra and its asymptotic counterpart. We then study the dynamics on the orbit i.e. the sphere S_2 which is the limit set for the trajectories of the discussed nonlinear systems. Such the dynamics is demonstrated to correspond to a generalization of the spherical pendulum involving the presence of the nonpotential forces. We identify the periodic solutions and show that the considered dynamical system on the sphere has the global bifurcation of a limit cycle from an equilibrium

mentioned above we conclude that this system has the global Hopf bifurcation.

2 The Newton-like equation

In this section we introduce the abstract Newton-like equation on a general Lie algebra. By choosing the concrete algebra this equation generates the nonlinear dynamical system satisfied by the time-dependent group parameters. Consider the following second-order differential equation on a Lie algebra $\mathfrak{g}$:

$$\mu\ddot{X} + \nu\dot{X} + \rho X + \sigma Y = e^{iX}Ye^{-iX}, \qquad X(0) = X_0, \quad \dot{X}(0) = \dot{X}_0, \tag{2.1}$$

where $X(t)\colon \mathbf{R} \to \mathfrak{g}$ is a curve in $\mathfrak{g}$, $Y \in \mathfrak{g}$ is a fixed element, $\mu > 0$, ν, ρ, $\sigma \in \mathbf{R}$ and dot designates differentiation with respect to time.

Demanding that (2.1) admits the solution on the orbit

$$\mathrm{Tr}X^2 = \mathrm{const} \neq 0, \tag{2.2}$$

assuming that μ, ν = const and ρ, σ are independent of Y, and using the differential consequence of (2.2) such that

$$\mathrm{Tr}X\ddot{X} = -\mathrm{Tr}\dot{X}^2, \tag{2.3}$$

we find

$$\sigma = 1, \qquad \rho = \mu\frac{\mathrm{Tr}\dot{X}^2}{\mathrm{Tr}X^2}. \tag{2.4}$$

Inserting (2.4) into (2.1), rescaling $t \to \sqrt{\mu}t$ and setting $\frac{\nu}{\sqrt{\mu}} = \beta$ we finally obtain the following Newton-like equation:

$$\ddot{X} + \beta\dot{X} + \frac{\mathrm{Tr}\dot{X}^2}{\mathrm{Tr}X^2}X = e^{iX}Ye^{-iX} - Y, \qquad X(0) = X_0, \quad \dot{X}(0) = \dot{X}_0. \tag{2.5}$$

We now discuss the asymptotic behaviour of (2.5). An immediate consequence of (2.5) is

$$\mathrm{Tr}X\ddot{X} + \beta\mathrm{Tr}X\dot{X} + \mathrm{Tr}\dot{X}^2 = 0. \tag{2.6}$$

Hence using

$$\mathrm{Tr}X\ddot{X} = \frac{d}{dt}\mathrm{Tr}X\dot{X} - \mathrm{Tr}\dot{X}^2, \tag{2.7}$$

we arrive at the following equation:

$$\frac{d^2}{dt^2}\mathrm{Tr}X^2 + \beta\frac{d}{dt}\mathrm{Tr}X^2 = 0, \tag{2.8}$$

$$\frac{d}{dt}\mathrm{Tr}X^2\Big|_{t=0} = 2\mathrm{Tr}X_0\dot{X}_0, \qquad \mathrm{Tr}X^2\Big|_{t=0} = \mathrm{Tr}X_0^2. \tag{2.9}$$

The solution of (2.8) is

$$\mathrm{Tr}X^2 = 2\mathrm{Tr}X_0\dot{X}_0\tfrac{1}{\beta}(1-e^{-\beta t}) + \mathrm{Tr}X_0^2. \tag{2.10}$$

It thus appears that whenever $\beta > 0$ then the solution to (2.5) approaches the orbit such that

$$\mathrm{Tr}X^2 = \tfrac{2}{\beta}\mathrm{Tr}X_0\dot{X}_0 + \mathrm{Tr}X_0^2. \tag{2.11}$$

In the case with $\beta = 0$ the solution (2.10) takes the form

$$\mathrm{Tr}X^2 = 2\mathrm{Tr}X_0\dot{X}_0 t + \mathrm{Tr}X_0^2, \tag{2.12}$$

i.e. there is no solution to (2.8) on the orbit unless

$$\mathrm{Tr}X_0\dot{X}_0 = 0, \tag{2.13}$$

that is the time evolution starts from the orbit which is then by virtue of (2.12) and (2.13) the invariant set. Clearly, the same holds true for $\beta < 0$.

Suppose that $\beta > 0$ and $\mathrm{Tr}X^2 \geq 0$ for all $X \in \mathfrak{g}$. As we have shown above the solution to (2.5) approaches then the orbit whose parameters depend via (2.11) on the initial data. Furthermore, it is clear that (2.5) has the asymptotics

$$\ddot{X} + \beta\dot{X} + \frac{\mathrm{Tr}\dot{X}^2}{r^2}X = \left(e^{iX}Ye^{-iX}\right)\Big|_{\mathrm{Tr}X^2=r^2} - Y, \tag{2.14}$$

$$\mathrm{Tr}X^2 = r^2, \tag{2.15}$$

where $r = \sqrt{\frac{2}{\beta}\mathrm{Tr}X_0\dot{X}_0 + \mathrm{Tr}X_0^2}$. Evidently, (2.15) can be regarded as a constraint. In section (4.2) (see also appendix) we demonstrate by an example of the $SU(2)$ group that the solutions of the unconstrained equation (2.14) of the form

$$\ddot{X} + \beta\dot{X} + \frac{\mathrm{Tr}\dot{X}^2}{r^2}X = \left(e^{iX}Ye^{-iX}\right)\Big|_{\mathrm{Tr}X^2=r^2} - Y, \tag{2.16}$$

where r is a constant, approach the orbit $\mathrm{Tr}X^2 = r^2$ regardless of the initial data, that is this orbit is a universal attracting set. The only exception are the initial conditions such that

$$X_0 = \mu Y, \qquad \dot{X}_0 = \nu Y, \qquad \mu,\, \nu \in \mathbf{R}. \tag{2.17}$$

These initial data correspond to the following ansatz for the solution to (2.16):

$$X(t) = x(t)Y, \tag{2.18}$$

$$\left[\ddot{x}+\beta\dot{x}+\frac{\mathrm{Tr}Y^2}{r^2}x\dot{x}^2\right]Y=\left[\left(e^{iX}Ye^{-iX}\right)\Big|_{\mathrm{Tr}X^2=r^2}\right]\Big|_{X=xY}-Y. \tag{2.19}$$

We note that (2.19) is not any asymptotics of the equation

$$\ddot{x}+\beta\dot{x}+\frac{\dot{x}^2}{x}=0, \tag{2.20}$$

arising from inserting the ansatz (2.18) into (2.5).

3 Periodic solutions to the Newton-like equation

This section is devoted to the periodic solutions to the abstract Newton-like equation (2.5). Consider eq. (2.5). Our aim is to study the solutions to (2.5) on the orbit, therefore we write (2.5) as

$$\ddot{X}+\beta\dot{X}+\frac{\mathrm{Tr}\dot{X}^2}{c}X=e^{iX}Ye^{-iX}-Y, \tag{3.1}$$

$$\mathrm{Tr}X^2=\mathrm{const}=c. \tag{3.2}$$

On using (3.1), (3.2) and the identity

$$\mathrm{Tr}(e^{iX}Ye^{-iX}-Y)^2=-2\mathrm{Tr}Y(e^{iX}Ye^{-iX}-Y), \tag{3.3}$$

we arrive at the following relation:

$$\mathrm{Tr}\ddot{X}^2+2\beta\mathrm{Tr}\dot{X}\ddot{X}+\beta^2\mathrm{Tr}\dot{X}^2-\frac{(\mathrm{Tr}\dot{X}^2)^2}{c}=-2\mathrm{Tr}Y(e^{iX}Ye^{-iX}-Y). \tag{3.4}$$

Furthermore, an immediate consequence of (3.1) is

$$\mathrm{Tr}Y\ddot{X}+\beta\mathrm{Tr}Y\dot{X}+\frac{\mathrm{Tr}\dot{X}^2}{c}\mathrm{Tr}YX=\mathrm{Tr}Y(e^{iX}Ye^{-iX}-Y). \tag{3.5}$$

Assuming that

$$\mathrm{Tr}YX=\mathrm{const}=\kappa, \tag{3.6}$$

we obtain from (3.5)

$$\frac{\kappa}{c}\mathrm{Tr}\dot{X}^2=\mathrm{Tr}Y(e^{iX}Ye^{-iX}-Y). \tag{3.7}$$

Inserting (3.7) into (3.4) we get

$$\mathrm{Tr}\ddot{X}^2+2\beta\mathrm{Tr}\dot{X}\ddot{X}+\left(\beta^2+\frac{2\kappa}{c}\right)\mathrm{Tr}\dot{X}^2-\frac{1}{c}(\mathrm{Tr}\dot{X}^2)^2=0. \tag{3.8}$$

$$\mathrm{Tr}Y^2$$

where $Y\neq 0$. Evidently, if (3.6) holds then

$$\dot V=\dot X, \tag{3.10}$$

and (3.8) can be written as

$$\mathrm{Tr}\ddot V^2+2\beta\mathrm{Tr}\dot V\ddot V+\left(\beta^2+\frac{2\kappa}{c}\right)\mathrm{Tr}\dot V^2-\frac{1}{c}(\mathrm{Tr}\dot V^2)^2=0. \tag{3.11}$$

Further, it follows immediately from (3.9) that

$$\mathrm{Tr}VY=0,\qquad \mathrm{Tr}V^{(\mathrm{n})}Y=0, \tag{3.12}$$

where $V^{(\mathrm{n})}$ is the n-*th* time derivative of V, and

$$\mathrm{Tr}V^2=\mathrm{Tr}X^2-\frac{(\mathrm{Tr}XY)^2}{\mathrm{Tr}Y^2}. \tag{3.13}$$

Clearly, whenever (3.2) and (3.6) take place then

$$\mathrm{Tr}V^2=\mathrm{const},\qquad \mathrm{Tr}V\dot V=0. \tag{3.14}$$

Suppose now (see (3.10)) that

$$\mathrm{Tr}\dot V^2=\mathrm{Tr}\dot X^2=\mathrm{const}=\tau, \tag{3.15}$$

and that the algebra $\mathfrak{g}$ is three dimensional, then in view of (3.12), (3.14) and (3.15) the triple $\{Y,\,V,\,\dot V\}$ forms the orthogonal basis of $\mathfrak{g}$. On expanding $\ddot V$ in this basis:

$$\ddot V=\mu V+\nu Y+\rho\dot V, \tag{3.16}$$

and using a differential consequence of (3.15) such that

$$\mathrm{Tr}\dot V\ddot V=0, \tag{3.17}$$

together with (3.12) we arrive at the following relation:

$$\ddot V=\mu V. \tag{3.18}$$

Hence using (3.14) one obtains

$$\mathrm{Tr}V\ddot V=-\mathrm{Tr}\dot V^2=\mu\mathrm{Tr}V^2. \tag{3.19}$$

Therefore,

$$\ddot V=-\frac{\mathrm{Tr}\dot V^2}{\mathrm{Tr}V^2}V, \tag{3.20}$$

$$\mathrm{Tr}V \quad = \quad \frac{\quad}{\mathrm{Tr}V^2}. \tag{3.21}$$

Finally, inserting (3.21) into (3.11) and taking into account (3.17) (see also (3.2), (3.6), (3.13) and (3.15)) we get

$$\tau\left[\tau\left(\frac{1}{c-\frac{\kappa^2}{\varepsilon}}-\frac{1}{c}\right)+\beta^2+\frac{2\kappa}{c}\right]=0, \tag{3.22}$$

where $\varepsilon = \mathrm{Tr}Y^2$. The validity of (3.22) is the condition for the existence of the solution to (2.5) satisfying (3.2), (3.6) and (3.15). In section 4.3 we show that in the case of the $SU(2)$ group such the solution corresponds to the periodic motion on the sphere S_2, more precisely, the uniform circular motion in a parallel. The counterpart of X is then the position vector in $\mathbf{R}^3$ and (3.20) is simply the normal acceleration. Notice that the case $\tau = 0$ corresponds to the equilibrium solutions to the equation (2.5) belonging to the orbit defined by (3.2). In fact, taking into account (3.7) and (3.15) when $\tau = 0$ as well as using (3.3) one finds

$$X = \lambda Y. \tag{3.23}$$

Hence, by virtue of (3.2) we get

$$\lambda = \pm\sqrt{\frac{c}{\varepsilon}}. \tag{3.24}$$

The formula (3.23) related to the ansatz (2.19) is also obtained in the case $c - \frac{\kappa^2}{\varepsilon} = 0$ which means that

$$\mathrm{Tr}X^2\mathrm{Tr}Y^2 = (\mathrm{Tr}XY)^2. \tag{3.25}$$

We end this section with the remark concerning equation (3.1), when $\beta = 0$ and $Y = 0$. Clearly, we then have

$$\ddot{X} = -\frac{\mathrm{Tr}\dot{X}^2}{\mathrm{Tr}X^2}X, \tag{3.26}$$

where $\mathrm{Tr}X^2 = \mathrm{const} = c$. Furthermore, an immediate consequence of (3.2) and (3.25) is

$$\mathrm{Tr}\dot{X}^2 = \mathrm{const}. \tag{3.27}$$

In the light of the interpretation of the formula (3.21) in the case of the $SU(2)$ group mentioned above it is plausible that then (3.26) corresponds to the geodesic equations on the sphere S_2. Obviously, the solutions to geodesic equations involve the periodic uniform motion in great circles.

4 Dynamics on the $SU(2)$ group

This section deals with the concrete realization of the abstract Newton-like equation (2.5) in the case of the $SU(2)$ group. We first derive the nonlinear dynamical system generated by (2.5) satisfied by the group parameters. Consider eq. (2.5). On taking into account the following realization:

$$\mathbf{J} = \tfrac{1}{2}\boldsymbol{\sigma}, \tag{4.1}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and σ_i, $i = 1, 2, 3$, are the Pauli matrices, of the infinitesimal generators J_i, $i = 1, 2, 3$, of the group $SO(3)$ which is locally isomorphic to $SU(2)$, we can write the general element of the Lie algebra $X(t)$ as

$$X(t) = \mathbf{x}(t)\cdot\frac{\boldsymbol{\sigma}}{2}, \tag{4.2}$$

where $\mathbf{x}(t)$: $\mathbf{R} \to \mathbf{R}^3$ and the dot designates the inner product. Analogously,

$$Y = \mathbf{a}\cdot\frac{\boldsymbol{\sigma}}{2}, \tag{4.3}$$

where $\mathbf{a}$ is a constant vector of $\mathbf{R}^3$. Now, inserting (4.2) and (4.3) into (2.5) and using the elementary properties of the Pauli matrices we arrive at the following nonlinear system of second-order differential equations:

$$\begin{aligned} &\ddot{\mathbf{x}} + \beta\dot{\mathbf{x}} + \frac{\dot{\mathbf{x}}^2}{\mathbf{x}^2}\mathbf{x} = (\cos|\mathbf{x}| - 1)\mathbf{a} + \frac{\sin|\mathbf{x}|}{|\mathbf{x}|}\mathbf{a}\times\mathbf{x} + (1 - \cos|\mathbf{x}|)\frac{(\mathbf{a}\cdot\mathbf{x})\mathbf{x}}{\mathbf{x}^2}, \\ &\mathbf{x}(0) = \mathbf{x}_0, \quad \dot{\mathbf{x}}(0) = \dot{\mathbf{x}}_0, \end{aligned} \tag{4.4}$$

where $\mathbf{a}\times\mathbf{x}$ designates the vector product of vectors $\mathbf{a}$ and $\mathbf{x}$, $|\mathbf{x}| = \sqrt{\mathbf{x}^2}$ stands for the norm of the vector $\mathbf{x}$ and $\dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt}$.

Notice that the orbits given by (2.2) are the two-dimensional spheres

$$\mathbf{x}^2 = \text{const}. \tag{4.5}$$

The linear stability analysis shows that the equilibrium $\overline{\mathbf{x}} = \mathbf{a}$, $\overline{\dot{\mathbf{x}}} = \mathbf{0}$ is unstable and the equilibrium $\overline{\mathbf{x}} = -\mathbf{a}$, $\overline{\dot{\mathbf{x}}} = \mathbf{0}$ is stable for sufficiently large $\beta > 0$.

We now discuss the asymptotic behaviour of the system (4.4). On making use of the relation

$$\mathrm{Tr}AB = \tfrac{1}{2}\mathbf{a}\cdot\mathbf{b} \tag{4.6}$$

for $A = \mathbf{a}\cdot\frac{\boldsymbol{\sigma}}{2}$ and $B = \mathbf{b}\cdot\frac{\boldsymbol{\sigma}}{2}$, and taking into account (4.2) and (4.3) we arrive at the following realization of (2.10) in the case of the $SU(2)$ group:

$$\mathbf{x}^2 = 2\mathbf{x}_0\cdot\dot{\mathbf{x}}_0\tfrac{1}{\beta}(1 - e^{-\beta t}) + \mathbf{x}_0^2. \tag{4.7}$$

$$2\mathbf{x}_0{\cdot}\dot{\mathbf{x}}_0 + \mathbf{x}_0^2 > 0, \tag{4.8}$$

and $\beta > 0$, then the solution to (4.4) approaches the orbit

$$\mathbf{x}^2 = \tfrac{2}{\beta}\mathbf{x}_0{\cdot}\dot{\mathbf{x}}_0 + \mathbf{x}_0^2. \tag{4.9}$$

Furthermore, it is clear that for the initial conditions such that

$$\mathbf{x}_0{\cdot}\dot{\mathbf{x}}_0 = 0, \tag{4.10}$$

and arbitrary β the orbit is an invariant set.

The remaining solutions do not approach any orbit. Namely, for $\beta > 0$ and

$$\tfrac{2}{\beta}\mathbf{x}_0{\cdot}\dot{\mathbf{x}}_0 + \mathbf{x}_0^2 = 0, \qquad \mathbf{x}_0{\cdot}\dot{\mathbf{x}}_0 \neq 0, \tag{4.11}$$

the solutions to (4.4) tend asymptotically to the singular point $\mathbf{x} = \mathbf{0}$. This point is approached after a finite period of time

$$t_* = -\frac{1}{\beta}\ln\left(1 + \frac{\beta\mathbf{x}_0^2}{2\mathbf{x}_0{\cdot}\dot{\mathbf{x}}_0}\right), \tag{4.12}$$

for the initial data satisfying the inequality

$$2\mathbf{x}_0{\cdot}\dot{\mathbf{x}}_0 + \mathbf{x}_0^2 < 0. \tag{4.13}$$

Finally, if $\beta \leq 0$ and $\mathbf{x}_0{\cdot}\dot{\mathbf{x}}_0 \neq 0$ then the trajectories go to infinity.

Let us now focus our attention on the most interesting case when $\beta > 0$ and the initial data satisfy the inequality (4.8) so that the solution to (4.4) approaches the sphere (4.9). We set for simplicity $\mathbf{a} = (0, 0, a_3)$. Based on the observations of section 4.3 discussing asymptotic dynamics on the orbit (see formula (4.44)) we find that whenever the following condition holds:

$$\beta^2\sqrt{2\beta^{-1}\mathbf{x}_0{\cdot}\dot{\mathbf{x}}_0 + \mathbf{x}_0^2} < |a_3|\left(1 + \cos\sqrt{2\beta^{-1}\mathbf{x}_0{\cdot}\dot{\mathbf{x}}_0 + \mathbf{x}_0^2}\right), \tag{4.14}$$

where $\beta > 0$ and

$$a_3\frac{\sin\sqrt{2\beta^{-1}\mathbf{x}_0{\cdot}\dot{\mathbf{x}}_0 + \mathbf{x}_0^2}}{\sqrt{2\beta^{-1}\mathbf{x}_0{\cdot}\dot{\mathbf{x}}_0 + \mathbf{x}_0^2}} \neq 0, \tag{4.15}$$

then the system (4.4) has the limit cycle given by (see (4.43) and (4.45))

$$x_3 = -\frac{\beta^2(2\beta^{-1}\mathbf{x}_0{\cdot}\dot{\mathbf{x}}_0 + \mathbf{x}_0^2)}{a_3\left(1 + \cos\sqrt{2\beta^{-1}\mathbf{x}_0{\cdot}\dot{\mathbf{x}}_0 + \mathbf{x}_0^2}\right)}, \tag{4.16}$$

$$x_1^2 + x_2^2 = R^2, \tag{4.17}$$

$$R=\sqrt{(2\beta^{-1}\mathbf{x}_0\cdot\dot{\mathbf{x}}_0+\mathbf{x}_0^2)\left[1-\frac{\beta^4(2\beta^{-1}\mathbf{x}_0\cdot\dot{\mathbf{x}}_0+\mathbf{x}_0^2)}{\left(1+\cos\sqrt{2\beta^{-1}\mathbf{x}_0\cdot\dot{\mathbf{x}}_0+\mathbf{x}_0^2}\right)^2 a_3^2}\right]}. \tag{4.18}$$

On the other hand, if the following inequality is valid (see (4.48)):

$$\beta^2\sqrt{2\beta^{-1}\mathbf{x}_0\cdot\dot{\mathbf{x}}_0+\mathbf{x}_0^2}>|a_3|\left(1+\cos\sqrt{2\beta^{-1}\mathbf{x}_0\cdot\dot{\mathbf{x}}_0+\mathbf{x}_0^2}\right), \tag{4.19}$$

then the trajectories tend to the equilibrium point such that

$$\overline{x}_1=0,\quad \overline{x}_2=0,\quad \overline{x}_3=-\mathrm{sgn}\,a_3\,\sqrt{2\beta^{-1}\mathbf{x}_0\cdot\dot{\mathbf{x}}_0+\mathbf{x}_0^2},\quad \overline{\dot{\mathbf{x}}}=\mathbf{0}, \tag{4.20}$$

where $\mathrm{sgn}\,x$ is the sign function.

It follows directly from (4.49) that the critical value of the bifurcation parameter β is the implicit function of the initial data given by the equation

$$\beta_c^2\sqrt{2\beta_c^{-1}\mathbf{x}_0\cdot\dot{\mathbf{x}}_0+\mathbf{x}_0^2}=|a_3|\left(1+\cos\sqrt{2\beta_c^{-1}\mathbf{x}_0\cdot\dot{\mathbf{x}}_0+\mathbf{x}_0^2}\right). \tag{4.21}$$

It should be noted that in view of (4.14)–(4.18) the system (4.4) has an infinite number of limit cycles. Evidently, these limit cycles are unstable.

4.2 Asymptotic dynamical system

We now study the nonlinear dynamical system derived from (2.17) for the $SU(2)$ group. In view of (4.4) such system is of the form

$$\ddot{\mathbf{x}}+\beta\dot{\mathbf{x}}+\frac{\dot{\mathbf{x}}^2}{r^2}\mathbf{x}=(\cos r-1)\mathbf{a}+\frac{\sin r}{r}\mathbf{a}\times\mathbf{x}+\frac{1-\cos r}{r^2}(\mathbf{a}\cdot\mathbf{x})\mathbf{x}. \tag{4.22}$$

As with (4.4) we find that the equilibrium $\overline{\mathbf{x}}=\pm r\frac{\mathbf{a}}{|\mathbf{a}|}$, $\overline{\dot{\mathbf{x}}}=\mathbf{0}$, with the plus sign is unstable and the equilibrium with minus sign is stable for large enough $\beta>0$.

We now discuss the $SU(2)$ realization of the equation (2.19). On setting

$$\mathbf{x}=x\mathbf{a}, \tag{4.23}$$

we obtain from (4.22) the following equation:

$$\ddot{x}+\beta\dot{x}+\frac{\mathbf{a}^2}{r^2}\dot{x}^2x=(1-\cos r)\left(\frac{\mathbf{a}^2}{r^2}x^2-1\right). \tag{4.24}$$

return to the asymptotic system (4.22). In appendix we show that whenever $\beta > 0$, then the solutions to (4.22) approach the orbit

$$\mathbf{x}^2 = r^2 \tag{4.25}$$

for arbitrary initial data. The only exception are the initial conditions of the form

$$\mathbf{x}_0 = \mu\mathbf{a}, \qquad \dot{\mathbf{x}}_0 = \nu\mathbf{a}, \qquad \mu,\, \nu \in \mathbf{R}, \tag{4.26}$$

corresponding to the case of the equation (4.24).

Suppose that $\beta > 0$. As with the system (4.4) we set for simplicity $\mathbf{a} = (0, 0, a_3)$. Suppose that the bifurcation parameter β satisfies the inequality

$$\beta < \sqrt{\frac{|a_3|(1 + \cos r)}{r}}, \tag{4.27}$$

where $a_3 \frac{\sin r}{r} \neq 0$. Using observations of section 4.3 we find that then the system (4.22) has the limit cycle of the form (see (4.43) and (4.45))

$$x_3 = -\frac{\beta^2 r^2}{a_3(1 + \cos r)}, \tag{4.28}$$

$$x_1^2 + x_2^2 = R^2, \tag{4.29}$$

where the radius of the limit cycle (the circle (4.29)) is

$$R = r\sqrt{1 - \frac{\beta^4 r^2}{(1 + \cos r)^2 a_3^2}}\,. \tag{4.30}$$

Further, for β such that

$$\beta > \sqrt{\frac{|a_3|(1 + \cos r)}{r}}, \tag{4.31}$$

the solutions to (4.22) approach the equilibrium point

$$\overline{x}_1 = 0, \quad \overline{x}_2 = 0, \quad \overline{x}_3 = -\operatorname{sgn} a_3\; r, \quad \dot{\overline{\mathbf{x}}} = \mathbf{0}. \tag{4.32}$$

Evidently, the critical value of the bifurcation parameter β is given by

$$\beta_c = \sqrt{\frac{|a_3|(1 + \cos r)}{r}}. \tag{4.33}$$

From the discussion completed in section 4.3 it follows that the system (4.22) has the global Hopf bifurcation at $\beta = \beta_c$. Phase portraits from numerical integration of the system (4.22) for $\beta > 0$ are illustrated in Figs. 1 and 2.

We now discuss the dynamics on the orbit, i.e. the two- dimensional sphere S_2 which has been already mentioned above to be attracting or invariant set for the investigated nonlinear dynamical systems for $\beta > 0$. In view of (4.22) the corresponding system can be written as

$$\ddot{\mathbf{n}} + \beta\dot{\mathbf{n}} + \frac{\dot{\mathbf{n}}^2}{r^2}\mathbf{n} = (\cos r - 1)\mathbf{a} + \frac{\sin r}{r}\mathbf{a}\times\mathbf{n} + \frac{1-\cos r}{r^2}(\mathbf{a}\cdot\mathbf{n})\mathbf{n}, \quad (4.34a)$$
$$\mathbf{n}^2 = r^2. \quad (4.34b)$$

Clearly, eq. (4.34b) is a constraint on (4.34a). By switching over to the spherical coordinates such that

$$\mathbf{n} = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta), \quad (4.35)$$

and setting (without loose of generality)

$$\mathbf{a} = (0, 0, a_3), \quad (4.36)$$

we obtain from (4.34) the following system:

$$\ddot{\theta} + \beta\dot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 - a_3\frac{1-\cos r}{r}\sin\theta = 0, \quad (4.37a)$$
$$\ddot{\varphi} + \beta\dot{\varphi} + 2\mathrm{ctg}\theta\,\dot{\theta}\dot{\varphi} - a_3\frac{\sin r}{r} = 0. \quad (4.37b)$$

We note that the system (4.37) is Lagrangian one. The Lagrangian is

$$L = \frac{1}{2}(\dot{\theta}^2 + \sin^2\theta\;\dot{\varphi}^2) - a_3\frac{1-\cos r}{r}\cos\theta, \quad (4.38)$$

and the system (4.37) can be written in the form of the Lagrange equations

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} &= Q_\theta,\\ \frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}} - \frac{\partial L}{\partial\varphi} &= Q_\varphi,\end{aligned} \quad (4.39)$$

where the generalized nonpotential forces are

$$Q_\theta = -\beta\dot{\theta}, \qquad Q_\varphi = -\beta\sin^2\theta\;\dot{\varphi} + a_3\frac{\sin r}{r}\sin^2\theta. \quad (4.40)$$

(4.39) as a generalized spherical pendulum. The standard spherical pendulum is easily seen to correspond to the particular case of $\beta = 0$ and $r = \pi$. The case $\beta = 0$ and $r = 2\pi$ or $\beta = 0$ and $a_3 = 0$ refers to the free motion on the sphere.

Consider the system (4.37). Let us assume that $a_3 \frac{\sin r}{r} \neq 0$. On setting $\theta =$ const, we find $\dot{\varphi} = \omega =$ const, so we then deal with the uniform circular motion in a parallel. Furthermore, we have

$$-\omega^2 = a_3 \frac{1 - \cos r}{r \cos\theta}, \tag{4.41a}$$

$$\beta\omega = a_3 \frac{\sin r}{r}. \tag{4.41b}$$

By virtue of (4.41a) $\theta \in (\frac{\pi}{2}, \pi]$ ("South of the equator") for $a_3 > 0$, and $\theta \in [0, \frac{\pi}{2})$ ("North of the equator") for $a_3 < 0$. The singular case $\theta = \frac{\pi}{2}$ can be treated as the motion in the equator with infinite angular velocity. In view of (4.41b) such a case corresponds to the limit $\beta = 0$.

We remark that the abstract formula (3.22) providing the criterion of periodic motion on the orbit takes the form

$$\omega^2 \cos^2\theta + \beta^2 + 2\frac{a_3 \cos\theta}{r} = 0, \tag{4.42a}$$

where $\omega = \dot{\varphi} =$ const and $\theta =$ const. On the other hand, the abstract relations (3.7) and (3.6) taken together yield

$$\omega^2 = -a_3 \frac{1 - \cos r}{r \cos\theta}, \tag{4.42b}$$

i.e. the formula (4.41a) is obtained. It can be easily checked that the system (4.41) is equivalent to the system (4.42).

We now return to the system (4.41). Let us assume that $\beta \neq 0$ and $r \neq \pi$, so we exclude the case of the standard spherical pendulum. An immediate consequence of (4.41) is the formula

$$\cos\theta = -\frac{\beta^2 r}{a_3(1 + \cos r)}. \tag{4.43}$$

Hence, recalling that the case of $\beta > 0$ is investigated, we find

$$\beta \leq \sqrt{\frac{|a_3|(1 + \cos r)}{r}}. \tag{4.44}$$

$$R = r\sqrt{1 - \frac{\beta^4 r^2}{(1+\cos r)^2 a_3^2}}. \tag{4.45}$$

We now come to the analysis of the asymptotic dynamics on the sphere S_2. Consider the system (4.37). Integrating (4.37b) yields

$$\dot\varphi = \left(a_3 \frac{\sin r}{r} \int\limits_0^t \sin^2\theta \; e^{\beta\tau} d\tau + \dot\varphi_0 \sin^2\theta_0 \right) \frac{e^{-\beta t}}{\sin^2\theta}\,. \tag{4.46}$$

We discuss the most interesting case of $\beta > 0$. Assuming that $\sin\theta \not\to 0$ we obtain from (4.46) the following asymptotic relation:

$$\dot\varphi = a_3 \frac{\sin r}{r} \lim_{t\to\infty} \frac{\int\limits_0^t \sin^2\theta \; e^{\beta\tau} d\tau}{\sin^2\theta \; e^{\beta t}} = a_3 \frac{\sin r}{r} \frac{1}{2\mathrm{ctg}\theta\,\dot\theta + \beta}\,. \tag{4.47}$$

Suppose now that $a_3\frac{\sin r}{r} \neq 0$. Inserting (4.47) into (4.37b) we find that at asymptotic times $\ddot\varphi = 0$, i.e. $\dot\varphi = \mathrm{const}$, which in view of (4.47) leads to $\theta = \mathrm{const}$. Clearly, the case $\sin\theta \to 0$ corresponds to approaching the equilibrium point $\theta = \pi$ for $a_3 > 0$ or $\theta = 0$ for $a_3 < 0$. Indeed, the equilibrium $\theta = 0$ for $a_3 > 0$ and $\theta = \pi$ for $a_3 < 0$ have been already mentioned in section 4.1 to be unstable. Taking into account (4.41) we see that whenever $\beta > 0$ and $a_3\frac{\sin r}{r} \neq 0$ then for β satisfying (4.44) the trajectories approach the circle given by (4.43). On the other hand, whenever β fulfils the inequality

$$\beta > \sqrt{\frac{|a_3|(1+\cos r)}{r}}, \tag{4.48}$$

then the trajectories tend to the equilibrium point $\theta = \pi$ for $a_3 > 0$ and $\theta = 0$ for $a_3 < 0$. Therefore, the critical value of the bifurcation parameter β is

$$\beta_c = \sqrt{\frac{|a_3|(1+\cos r)}{r}}. \tag{4.49}$$

Notice that for $\beta = \beta_c$ the radius of the circle given by (4.45) equals zero, that is the periodic trajectory (limit cycle) reduces to the equilibrium point $\theta = \pi$ ($\theta = 0$). Furthermore, as we have shown above the limit cycle given by (4.43) and the equilibrium point $\theta = \pi$ ($\theta = 0$) are universal attracting sets for $0 < \beta < \beta_c$ and $\beta > \beta_c$, respectively. We conclude that whenever $a_3\frac{\sin r}{r} \neq 0$ and $\beta > 0$ then the system (4.37) has a global Hopf bifurcation (the bifurcation of a limit cycle from an equilibrium point) at $\beta = \beta_c$.

find that the system (4.37) is asymptotically Hamiltonian one.

5 Conclusion

In this work we have introduced an abstract second-order Newton-like differential equation on a general Lie algebra such that orbits of the Lie-group action are attracting set. By passing to the group parameters corresponding to the concrete Lie algebra the Newton-like equation generates the nonlinear dynamical system referred to as the group-invariant one with the attracting set coinciding with the orbit. In other words, a method has been found in this paper for constructing nonlinear systems with the presumed form of an attracting set. We have demonstrated that in the case of the $SU(2)$ group the attracting manifold is the universal attracting set for the system generated by the asymptotic version of the abstract Newton-like equation. It is suggested that such universality holds true, at least in some regions of a phase space, for general Lie groups. In order to clarify the asymptotic behaviour of the nonlinear dynamical systems arising in the case with the $SU(2)$ group we have studied the dynamical system on the sphere S_2 corresponding to the generalized spherical pendulum. Such system has been shown to have the global Hopf bifurcation. Based on this observation we have found that the group-invariant dynamical system is an interesting example of the system which has an infinite number of unstable limit cycles. On the other hand, it has turned out that the asymptotic dynamical system has a global Hopf bifurcation. We note that the discussed limit cycles (circles) arising in the case of the $SU(2)$ group coincide with sections of the sphere S_2 by the planes perpendicular to the vector $\mathbf{a}$. In our opinion, such a coincidence takes place in the general case of an orientable two-dimensional Lie-group manifold, that is the concrete form of the limit cycles (whenever they exist) is determined by the orbit of the Lie-group action and the direction of the element Y. Thus we suggest that the actual treatment provide a method for constructing nonlinear dynamical systems with the presumed form of a limit cycle. The example of the $SU(2)$ group discussed herein with simple oscillatory dynamics on the orbit is a model one. We recall that numerous group-theoretic constructions are tested by the example of the $SU(2)$ group. It seems that in the case of groups with higher-dimensional orbits the dynamics shoul be much more complex. Since we do not know any criterion excluding symmetry of the chaotic attractor, therefore we should admit that the dynamics would be chaotic one. We also observe that even in the case with the two-dimensional orbits some open questions arise when the orbit is not a connected set. For example, it is not clear referring to the experience with the $SU(2)$ group what is the behaviour of the system having the orbit

of the above comments it seems that the group-theoretic approach introduced herein would be a useful tool in the theory of nonlinear dynamical systems, especially in the bifurcation theory, as well as in such branches of applied mathematics as for example the control theory.

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We now demonstrate that the solutions to the system (4.22) with $\beta > 0$ approach the sphere $\mathbf{x}^2 = r^2$ for arbitrary initial data except those ones satisfying the condition (4.26) referring to the case of the equation (4.24). Consider the system (4.22). We assume for simplicity (without loose of generality) that $\mathbf{a} = (0, 0, a_3)$. By switching over to the spherical coordinates such that

$$\begin{aligned} x_1 &= \rho \sin\theta \cos\varphi, \\ x_2 &= \rho \sin\theta \sin\varphi, \\ x_3 &= \rho \cos\theta, \end{aligned} \tag{A.1}$$

we arrive at the following system:

$$\ddot{\rho} + \beta\dot{\rho} + \rho(\dot{\theta}^2 + \sin^2\theta\ \dot{\varphi}^2)\left(\frac{\rho^2}{r^2} - 1\right) + \dot{\rho}^2\frac{\rho}{r^2} = a_3(1 - \cos r)\cos\theta\left(\frac{\rho^2}{r^2} + 1\right), \tag{A.2a}$$

$$\ddot{\theta} + \beta\dot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 + \frac{2\dot{\rho}}{\rho}\dot{\theta} - a_3\frac{1 - \cos r}{\rho}\sin\theta = 0, \tag{A.2b}$$

$$\ddot{\varphi} + \beta\dot{\varphi} + 2\mathrm{ctg}\theta\,\dot{\theta}\dot{\varphi} + \frac{2\dot{\rho}}{\rho}\dot{\varphi} - a_3\frac{\sin r}{r} = 0. \tag{A.2c}$$

Integration of (A.2c) yields

$$\dot{\varphi} = \frac{\left(a_3\frac{\sin r}{r}\int\limits_0^t \rho^2\sin^2\theta\ e^{\beta\tau}d\tau + \dot{\varphi}_0\rho_0^2\sin^2\theta_0\right)e^{-\beta t}}{\rho^2\sin^2\theta}. \tag{A.3}$$

Suppose that $\beta > 0$ and $\sin\theta \not\to 0$. We also assume that $a_3\frac{\sin r}{r} \neq 0$. Notice that this inequality implies $\rho \not\to 0$ since then $\mathbf{x} \equiv 0$ is not a solution of (4.22). Proceeding analogously as with eq. (4.46) we arrive at the following asymptotic relation:

$$\dot{\varphi} = a_3\frac{\sin r}{r}\frac{1}{\frac{2\dot{\rho}}{\rho} + 2\mathrm{ctg}\theta\,\dot{\theta} + \beta}. \tag{A.4}$$

In view of (A.2c) this leads to $\ddot{\varphi} = 0$, i.e. $\dot{\varphi} = \mathrm{const}$. Hence we find that at asymptotic times

$$\frac{\dot{\rho}}{\rho} + \mathrm{ctg}\theta\,\dot{\theta} = \mathrm{const}. \tag{A.5}$$

Integrating (A.5) we obtain

$$\rho\sin\theta = Ce^{\mu t}, \tag{A.6}$$

where C and μ are constant. Suppose that $\mu > 0$. By virtue of (A.6) this implies $\rho \to \infty$. Hence, taking into account (A.2a)–(A.2c) and differential

$$- \beta\mu - \frac{\dot{\rho}^2}{r^2} - \frac{\mathrm{tg}^2\theta\,(\mu\rho - \dot{\rho})^2}{r^2} - \frac{\rho^2\sin^2\theta\;\dot{\varphi}^2}{r^2} + a_3\frac{1-\cos r}{r^2}\rho\cos\theta = \mu^2. \quad \text{(A.7)}$$

But $\frac{\rho^2\sin^2\theta\;\dot{\varphi}^2}{r^2} \gg a_3\frac{1-\cos r}{r^2}\rho\cos\theta$. Therefore, the left-hand side of (A.7) is negative and the right-hand side is positive. This contradiction shows that μ in (A.6) cannot be positive. Let us assume now that $\mu < 0$. Then $\rho \to 0$ and in view of (A.1) $\mathbf{x} \to 0$. But $\mathbf{x} = 0$ is not the solution to (4.22) when $a_3\frac{\sin r}{r} \neq 0$. Thus it turns out that $\mu = 0$ and the asymptotic formula (A.6) reduces to

$$\rho\sin\theta = C. \quad \text{(A.8)}$$

Now, eqs. (A.2a)–(A.2c) and (A.8) taken together yield

$$-\frac{(\rho\dot{\rho})^2}{\rho^2\cos^2\theta} + (r^2 - \rho^2\sin^2\theta)\dot{\varphi}^2 + a_3(1-\cos r)\rho\cos\theta = 0. \quad \text{(A.9)}$$

On integrating (A.9) written with the help of (A.8) in the form independent of θ, one can easily find that $\dot{\rho} \neq 0$ at asymptotic times, leads to contradiction. Therefore $\dot{\rho} = 0$, i.e. $\rho = \text{const}$. Hence by virtue of (A.8) we have $\theta = \text{const}$. Assuming that $\rho \neq r$, using (A.2a) and (A.2b), and taking into account that $\rho \neq 0$, $\sin\theta \neq 0$, $\dot{\varphi}^2 \neq 0$, we get $\rho\dot{\varphi}^2 = 0$, a contradiction. Thus we find $\rho = r$. Notice that an immediate consequence of this observation and eq. (A.9) is the formula (4.41a) describing the uniform circular motion on the sphere $\mathbf{x}^2 = r^2$. Clearly, the case $\sin\theta \to 0$ corresponds to approaching the stable equilibrium $\overline{\mathbf{x}} = (0, 0, -\mathrm{sgn}\, a_3\; r)$, $\dot{\overline{\mathbf{x}}} = \mathbf{0}$ of (4.22). We finally note that in the case of the ansatz (4.23) implying eq. (4.24) we have $\sin\theta \equiv 0$ in (A.1) and the system (A.2) becomes singular one.

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Fig. 1. The system (4.22) with $\beta = 0.1$, $r = 1$, $\mathbf{a} = (0, 0, 1)$ and $\mathbf{x}_0 = (1.4, 1, 1)$, $\dot{\mathbf{x}}_0 = (0.1, 0.1, 0.1)$. The parameter β satisfies (4.27). The initial condition fulfils $\mathbf{x}_0^2 > r^2$. The trajectory starts from the point marked with the cross. Left: the projection on the (x_1, x_2) plane. Right: the projection on the (x_1, x_3) plane.

Fig. 2. The system (4.22), where β, r, $\mathbf{a}$, are the same as in Fig. 1, and $\mathbf{x}_0 = (0.1, 0.1, 0.1)$, $\dot{\mathbf{x}}_0 = (0.5, 0.5, 0.5)$. The initial data satisfy $\mathbf{x}_0^2 < r^2$. Left: the projection on the (x_1, x_2) plane. Right: the projection on the (x_1, x_3) plane.